FAST HILBERT TRANSFORM

by

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ABSTRACT

The use of Hilbert transform is becoming more and more important for analysis and processing of geophysical data. However, the direct mathematical formulation in the form of contour integration is not easy to programme.

A specific formulation which relates the Hilbert transform and the Fourier transform has been established for developing a computer programme. This relationship enables us to execute the Hilbert transformation in a very quick manner using the well known Fast Fourier transform algorithm.

The aplication of this method for generating quadrature seismic trace and recovering minimum phase spectrum from the magnitude demonstrates the effectiveness of the programme. The conversion of non-minimum phase seismic wavelet into its corresponding minimum phase wavelet which has similar spectral magnitude can be done using the Hilbert transform.

I. INTRODUCTION

Although the mathematical concept was introduced more than 60 years ago (Titchmarch, 1926, 1937, 1948), the use of Hilbert transform in the search of oil and gas has just been introduced recently (Taner et al, 1979; Robertson and Nogami, 1984). In this case the Hilbert transform is applied to generate the attributes of the complex seismic data.

The Hilbert transform not only does contribute in the analysis of the complex seismic trace but also gives a valuable contribution to the analysis of gravity and magnetic data (Shuey, 1972; Nabighian, 1972; Cerveny and Zahradnick, 1975; Stanley and; Green, 1976; Stanley 1977; Mohan et al, 1982). In this paper the use of the Hilbert transform only for the analysis of seismic data will be discussed. Implementation of computer programme for fast Hilbert transform will also be discussed.

In seismic data analysis, the Hilbert transform is used for generating the quadrature function of a seismic trace. The quadrature trace can be considered as the potential energy as a pair of the kinetic energy recorded by geophones. As a result from the passage of seismic waves, the geophones record the velocity of the particle motion which manifest as the amplitude of the seismic trace. The square of the velocity is proportional to the kinetic energy. On the contrary, as the particle motion is opposed by an elastic restoring force of the rock, the energy become stored as potential energy (Taner et al, 1979).

An advantage of having the quadrature trace is that it can be combine with the real seismic trace to form an anvelope of the seismic waves which is referred to as the reflection strength, The reflection strength is proportional to the square root of the total energy. A portion of a real seismic trace, the corresponding quadrature trace and the reflection strength is given in Figure 1. Apart from the reflection strength, the instantaneous frequency can also be generated.

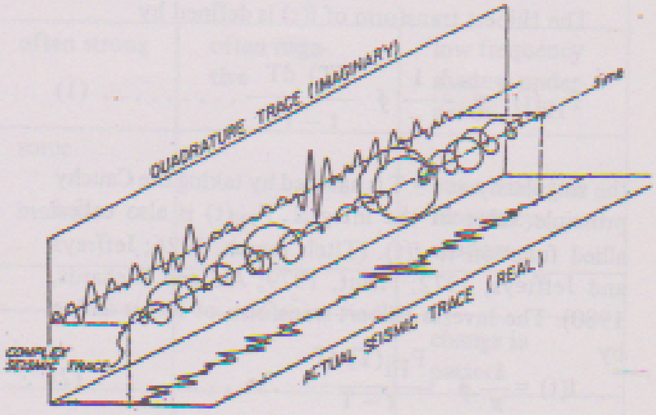

Figure 1. A portion of a real seismic trace, its corresponding quadrature trace and the reflection strength. (from Taner et al, 1979)

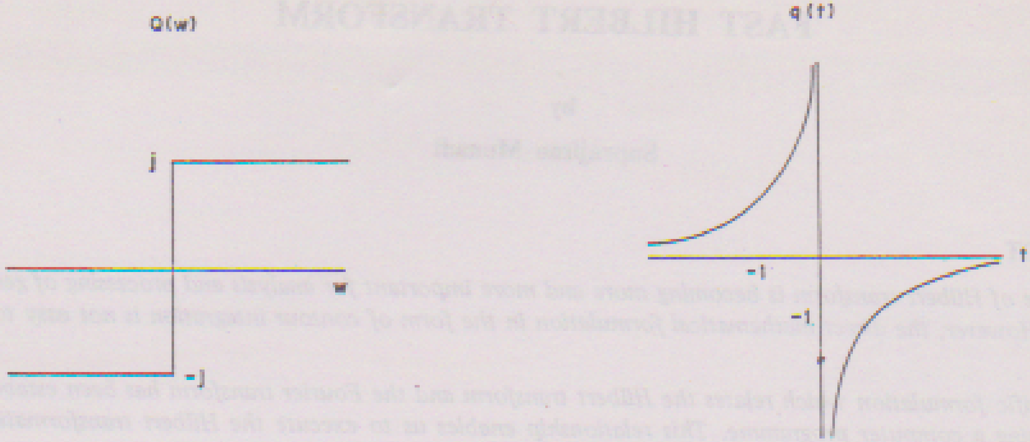

Figure 2. The sgn ω as a function of frequency and its Fourier transform, the Kernel function as a function of time.

The maximum points of the reflection strength differs from the maximum points of peak or through amplitude, especially when the reflection is composed by interference of several sub-reflections. In this case the reflection strength has its maximum at phase point (Taner et al, 1979). Polarity is derived at the maximum points of the reflection strength. The geological indications by reflection strength, instantaneous phase, instantaneous frequency and polarity is summarized in Table -1.

II. REVIEW OF MATHEMATICAL BACKGROUND

A. *Basic formula*

The Hilbert transform of f(t) is defined by

$$F_{Hi}(t) = -\frac{1}{\pi} \oint \frac{f(T)\ dT}{t - T} \quad \ldots\ldots\ldots \quad (1)$$

the singularity at t = T is handled by taking the Cauchy principle value of the integral. $F_{Hi}(t)$ is also called allied function to f(t). (Titch march, 1926; Jeffreys and Jeffreys, 1972; Pilant, 1979; Aki and Richards, 1980). The inverse Hilbert transform of f(t) is define by

$$f(t) = \frac{1}{\pi} \oint \frac{F_{Hi}(T)\, dT}{t - T} \quad \ldots\ldots\ldots\ldots \quad (2)$$

It can be shown (Bracewell, 1965) that $F_{Hi}(t)$ is a linear functional of f(t). This relationship can be expressed as (see also Aki and Richards, 1980; Kanasewich, 1975, 1981).

$$F_{Hi}(t) = -\frac{1}{\pi t} * f(t) \quad \ldots\ldots\ldots\ldots \quad (3)$$

where $(-\pi t)^{-1}$ is the Kernel function whose Fourier transform is j sgn ω which is equal to +j for positive ω and -j for negative ω. (see Figure 2).

* denote convolution.

The Hilbert transform of a Kernel function is the Dirac delta function.

Figure 3 is an example of Hilbert transform in succession (Bracewell, 1965). The left-hand side shows a function and the result of two successive Hilbert transformation, while the right-hand column shows the corresponding Fourier transforms. It can be seen that the Hilbert transform conserves the spectral amplitude but alteres the phase by 90 degrees, positively or negatively according to the sign of ω. It can also be seen that the Hilbert transform of even functions are odd and those of odd function even.

Equation (3) shows that the Hilbert transform can be found by Fourier transforming f(t) to yield F(ω), and the 90 degrees phase advance can be carried out by interchanging real and imaginary part of F(ω), with sign change in the resulting imaginary part. An inverse Fourier transform then returns to the required Hilbert transform (Aki and Richards, 1980).

Geological Indications by Seismic Parameters

	Phase	Reflection Strength	Polarity	Frequency
STRUCTURAL FEATURES				
Correlation	emphasizes Continuity of Weak events			
Unconformities	angularities show well	often strong and variable		
Faults	show clearly	pattern helps correlation across fault		
STRATIGRAPHIC FEATURES				
Major breaks in section	show on-lap, off-lap patterns	often strong; nature of contrast may indicate clastic to carbonate or vice-versa		moderately low frequency
Pinchouts	often show clearly	variation in pattern		
Prograding	often show clearly			
Turbidites	local irregular mound pattern			
Reefs	local pattern	interruption in patterns, lower frequency		
FLUID CONTENT				
Gas		often strong	often negative	low frequency shadow underneath
Condensate		some increase		sometimes low frequency shadow
Flat Spot	shows clearly			
Limits to Production				change in pattern
Distinguish gas from lime buildup, conglomerate, etc.		sometimes distinctive	sometimes distinctive	

Table - 1. (from Taner et al, 1979)

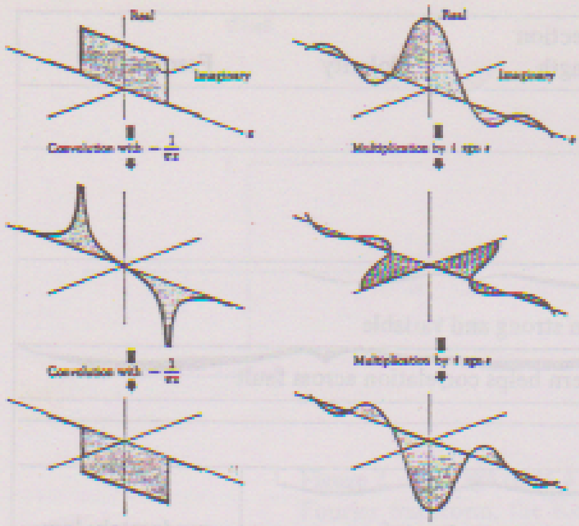

Figure 3. The left-hand column shows a function and the result of two successive Hilbert transformations, while the right-hand column shows the corresponding Fourier transforms. (from Bracewell, 1965)

B. Analytic signal and Hilbert transform

A function is called analytic if it is continous in the time domain and the frequency response is zero for negative frequencies. Let

$$y(t) = f(t) - j\ F_{Hi}(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

be an analytic function. $F_{Hi}(t)$ is referred to as the quadrature function of f(t).

For the sake of clarity, let us consider an analytic function of the form

$$\exp(-jt) = \cos(t) - j\ \sin(t) \quad \ldots\ldots\ldots\ldots (5)$$

The analytic signal corresponding to cos(t) is exp(jt) and the Hilbert transform of cos(t) is – sin(t). Thus, by analogy it can be conluded that the analytic signal bears the same relationship to f(t) as exp(–jt) does to cos(t) (Bracewell, 1965).

C. Hilbert transform of a causal function

An infinite y(t) is causal if y(t) = 0 for $t<0$. Kanasewich (1965, 1981) has shown that for a linear system which is physically realizable whose impulse response is causal, the real and imaginary part are Hilbert transform.

$$Y_R(\omega) = \frac{1}{\pi} \oint \frac{Y_I(w)}{\omega - w}\, dw \quad \ldots\ldots\ldots (6)$$

$$Y_I(\omega) = -\frac{1}{\pi} \oint \frac{Y_R(w)}{\omega - w}\, dw \quad \ldots\ldots (7)$$

Where :

$Y_R(\omega)$ is the real part of the spectra

$Y_I(\omega)$ is the imaginary part of the spectra

Equation (6) and (7) are comparable to the log magnitude spectra and phase spectra of a causal function.

Let

$$Y(\omega) = [\ Y(\omega)\ ]\ e^{j\,\phi(\omega)} \quad \ldots\ldots\ldots\ldots (8)$$

The log modulus and the phase are Hilbert transforms

$$\log[\ Y(\omega)\ ] = -\frac{1}{\pi} \oint \frac{\phi(\omega)}{w - \omega}\, dw \quad \ldots\ldots (9)$$

$$\phi(\omega) = \frac{1}{\pi} \oint \frac{\log[\ y(w)\ ]}{w - \omega}\, dw \quad \ldots\ldots\ldots (10)$$

Interested reader can find proof of these equation in Oppenheim and Schafer (1975).

Equations (9) and (10) tell us that we can obtain the phase spectrum of any signal and any instrument by only knowing their spectral modulus. This is important if we want to correct the phase caused by the instrument which is generally not known.

Another interesting application of equations (9) and (10) is the conversion of a non minimum phase wavelet into its corresponding minimum phase wavelet. The corresponding minimum phase wavelet has the same spectral magnitude as the minimum phase wavelet. This application arises from the fact that the Hilbert transform pair given by equation (9) and (10) are derived from minimum phase assumption, so that the phase obtained from equation (10) is the minimum version of the phase spectrum of the signal whose spectral magnitude is Y (ω).

D. Reflection strength, instantaneous phase and instantaneous frequency

Let us refer to the analytic function given by equation (4), the reflection strength or the envelope is defined by

$$E(t)=[f^2(t)+F^2_{Hi}(t)]^{1/2} \quad \ldots\ldots\ldots\ldots (11)$$

from which the instantaneous phase can be found as

$$\theta(t) = \tan^{-1}[F_{Hi}(t)/f(t)] \quad \ldots\ldots\ldots (12)$$

Since

$$\theta(t) = \omega t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

where ω is the angular frequency, the instantaneous frequency can be found by differentiating the instantaneous phase with respect to time. The differentiation of a function can be carried out easily in the frequency domain by multiplying its spectra by $j\omega$. The instantaneous frequency maybe useful function to plot for data that shows dispersion (Kanasewich, 1981).

III. IMPLEMENTATION PROCEDURES

The fast Hilbert transform can be carried out by using the Fast Fourier transform. For this purpose, the relationship between both transforms must be established. It should be noted that the Fast Fourier transform operates on the discrete, finite duration sequence of data.

Kanasewich (1975, 1981) has shown that

$$F_H(\omega) = j\,F(\omega)\,\mathrm{sgn}\,\omega \quad \ldots\ldots\ldots\ldots (14)$$

which is just the Fourier transform of the equation (3) and

$$\mathrm{Sgn}\,\omega = \begin{cases} 1 & \text{for } \omega > 0 \\ 0 & \omega = 0 \\ -1 & \omega < 0 \end{cases} \quad \ldots\ldots\ldots (15)$$

Let rewrite analytic function given by equation (4) in the form :

$$z(t) = f(t) - j\,f_H(t) \quad \ldots\ldots\ldots\ldots\ldots (16)$$

its Fourier transform can be written as

$$Z(\omega) = F(\omega) - j\,F_H(\omega) \quad \ldots\ldots\ldots\ldots (17)$$

Now substitute equation (15) into (14) and combine with equation (17) to yield :

$$\begin{aligned} Z(\omega) &= F(\omega) - (-F(\omega)) && \text{for } \omega > 0 \\ &= F(\omega) - 0 && \text{for } \omega = 0 \\ &= F(\omega) - F(\omega) && \text{for } \omega < 0 \end{aligned}$$

Which means that the spectrum of an analytic function is related to the spectrum of its real function by :

$$\begin{aligned} Z(\omega) &= 2F(\omega) && \text{for } \omega > 0 \\ &= F(\omega) && \omega = 0 \\ &= 0 && \omega < 0 \end{aligned} \qquad (18)$$

Inverse Fourier Transform of equation (18) yields an analytic function in which its real and imaginary part are Hilbert transform pair.

By considering equations (16) and (18), the Hilbert transform can be executed using the Fast Fourier transform through the following steps :

1. Fourier transforming $f(t)$ to yield $F(\omega)$.
2. Set $Z(\omega) = 2F(\omega)$ for $n = 2$ to $\frac{N}{2}$

   (n is the number of sample in frequency domain).
3. Set $Z(\omega) = F(\omega)$ for $n = 1$.
4. Set $Z(\omega) = 0$ for $n = \frac{N}{2} + 1$ to N.
5. Inverse Fourier transforming $Z(\omega)$ from $n = 1, N$ to yield a real and an imaginary function in the time domain.
6. The negative of the imaginary part of step 5 is the Hilbert transform of its real part and vice versa.

IV. EXPERIMENTAL RESULTS AND DISCUSSION

A. *Conversion of maximum phase wavelet into maximum phase correspondence*

A minimum phase doublet of amplitude 2,1 and a maximum phase doublet of amplitude 1,2 has exactly the same spectral magnitude as shown in Figure 4-a. Their amplitude phase are different. The phase spectrum of the minimum phase wavelet is illustrated in Figure 4-b, while the phase spectrum of the maximum delay wavelet is given in Figure 4-c.

A computer programme has been written to take the Hilbert transform of the log spectral magnitude

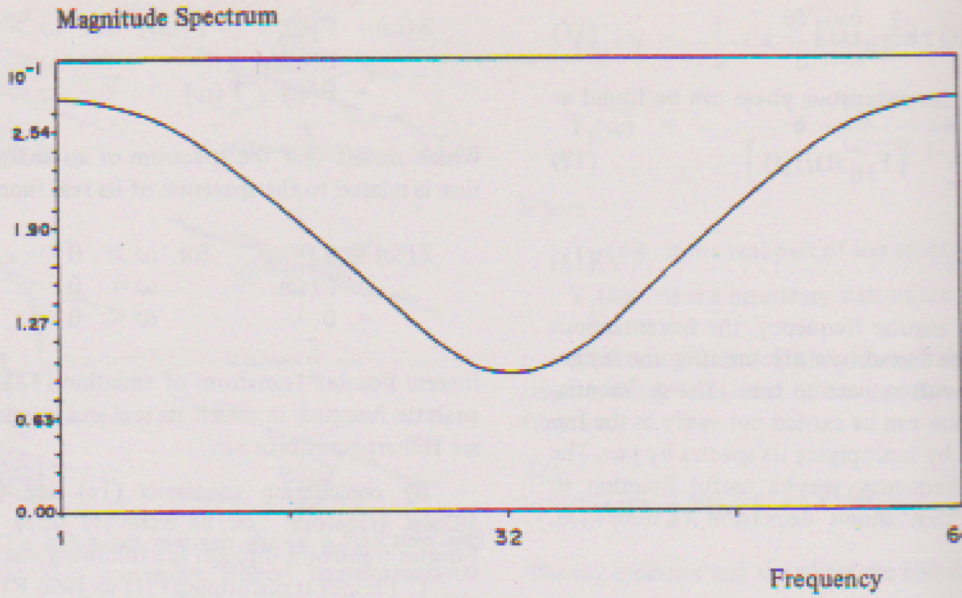

Figure 4–a. The magnitude spectrum of a 2,1 doublet **and a 1,2 doublet. Their spectral magnitude are exactly the same. Frequency must be multiplied by 15.625 Hz for all magnitude and phase spectra in this paper.**

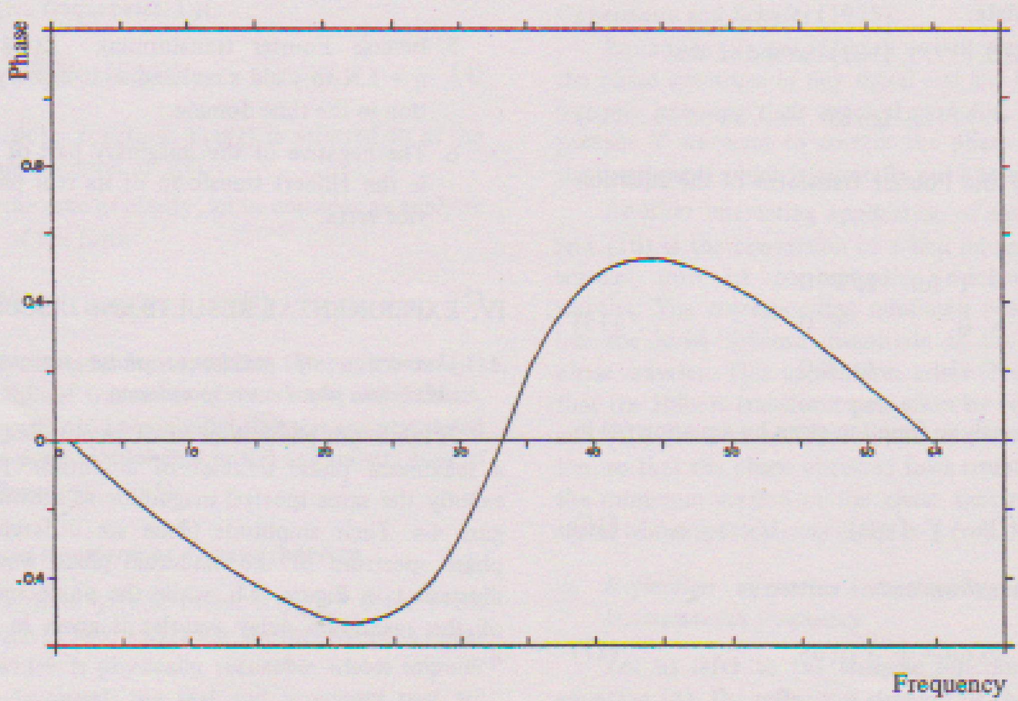

Figure 4–b. The phase spectrum of the 2, 1 wavelet.

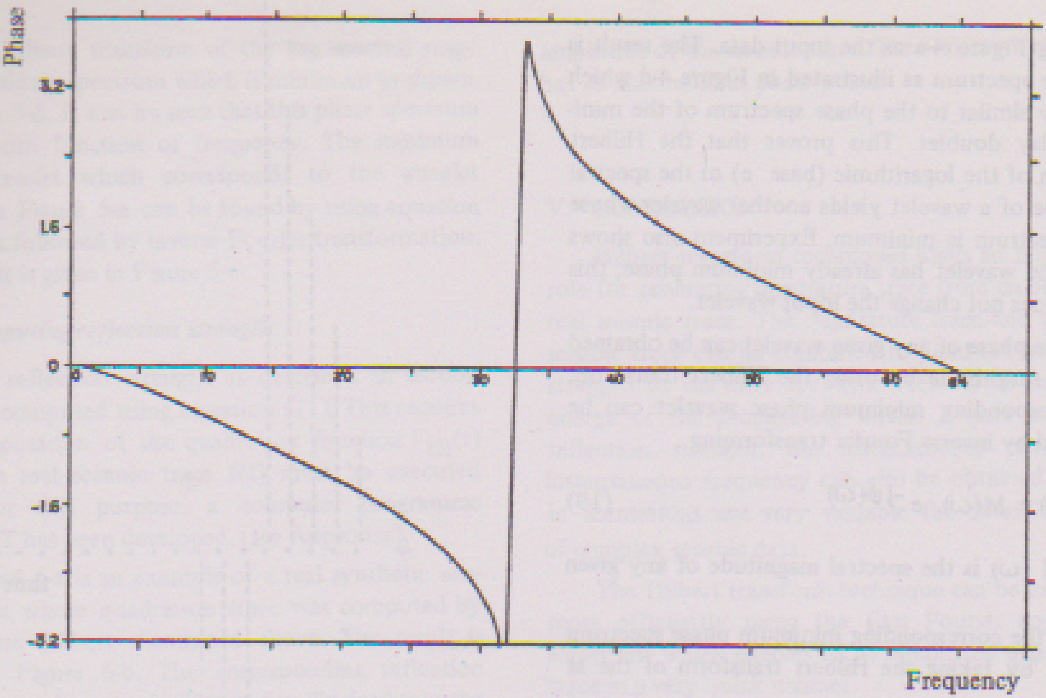

Figure 4–c. The phase spectrum of the 1,2 wavelet.

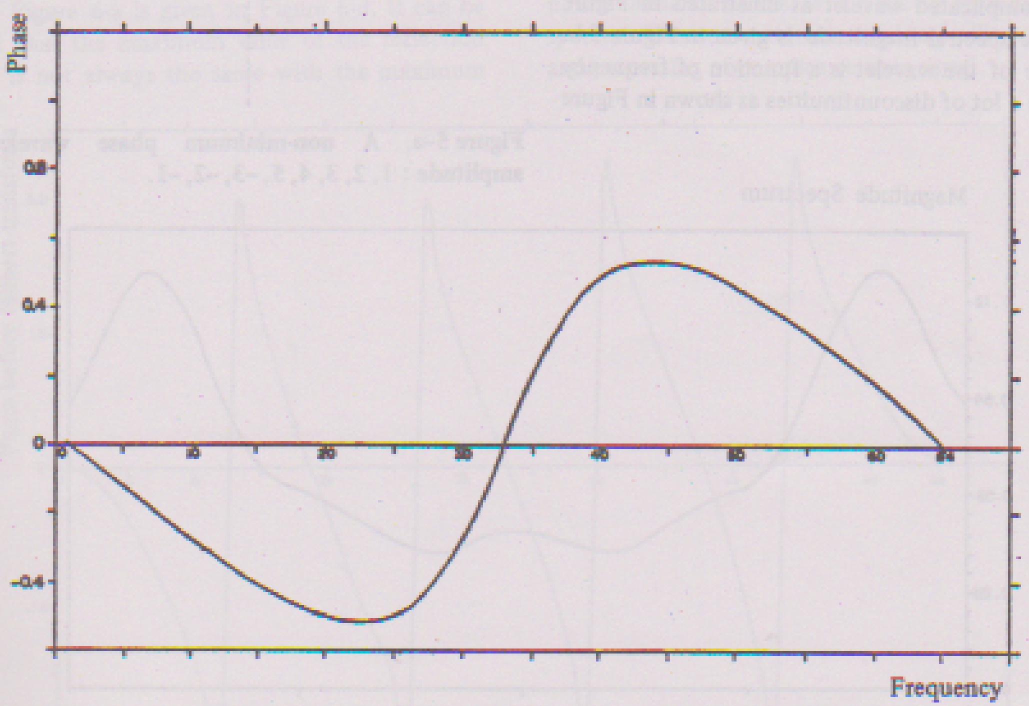

Figure 4–d. The phase spectrum of the 1,2 wavelet after Hilbert transforming the log spectral magnitude of Figure 4–a.

and using Figure 4-a as the input data. The result is the phase spectrum as illustrated in Figure 4-d which is exactly similar to the phase spectrum of the minimum delay doublet. This proves that the Hilbert transform of the logarithmic (base e) of the spectral magnitude of a wavelet yields another wavelet whose phase spectrum is minimum. Experiment also shows that if the wavelet has already minimum phase, this process does not change the input wavelet.

If the phase of any given wavelet can be obtained from its magnitude by using the Hilbert transform, the corresponding minimum phase wavelet can be computed by inverse Fourier transforming

$$S(\omega) = M(\omega).\ e^{-j\phi(\omega)} \quad \ldots\ldots\ldots\ldots \quad (19)$$

Where $M(\omega)$ is the spectral magnitude of any given wavelet.
$\phi(\omega)$ is the corresponding minimum phase spectrum obtained by taking the Hilbert transform of the $M(\omega)$.

Another interesting example is the conversion of a more complicated wavelet as illustrated in Figure 5-a. Whose spectral magnitude is given in Figure 5-b. The phase of the wavelet is a function of frequency which has a lot of discountinuities as shown in Figure 5-c

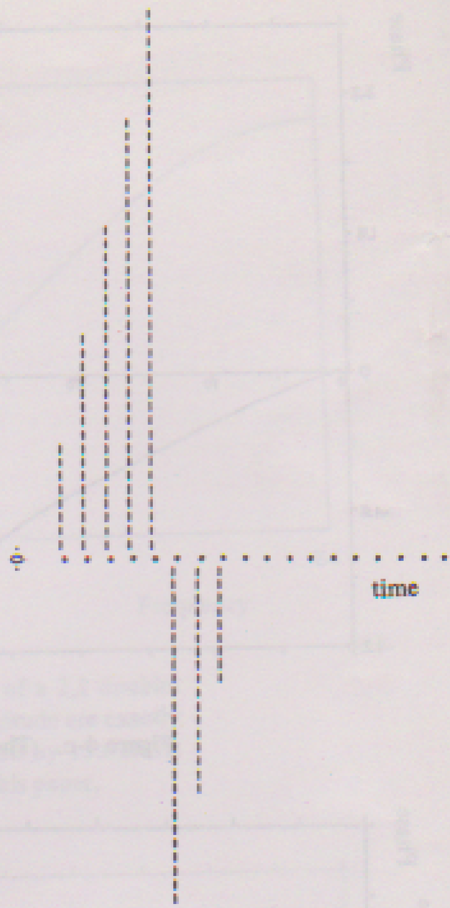

Figure 5-a. A non-minimum phase wavelet with amplitude : 1, 2, 3, 4, 5, -3, -2, -1.

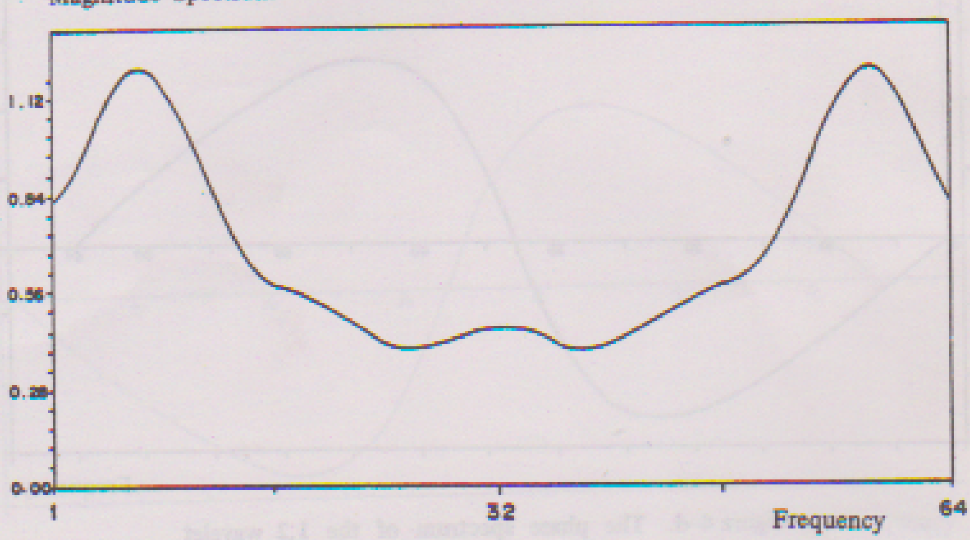

Figure 5-b. The magnitude spectra of Figure 5-a.

The Hilbert transform of the log spectral magnitude yields a spectrum which is minimum as shown in Figure 5-d. It can be seen that this phase spectrum is a smooth function of frequency. The minimum phase wavelet which corresponds to the wavelet shown in Figure 5-a can be found by using equation (19) then followed by inverse Fourier transformation. The result is given in Figure 5-e.

B. Computing reflection strength

The reflection strength as described in section II.4 was computed using equation (11). This requires the computation of the quadrature function $F_{Hi}(t)$ from the real seismic trace $f(t)$ must be executed first. For this purpose, a computer programme HILBERT has been developed, (see Appendix).

Figure 6-a is an example of a real synthetic seismic trace whose quadrature trace was computed by subroutine Hilbert mentioned above. The result is given in Figure 6-b. The corresponding reflection strength can be seen in Figure 6-c. To facilitate the comparation between the reflection strength and the absolute value of the trace amplitude, the absolute value of Figure 6-a is given in Figure 6-d. It can be observed that the maximum value of the reflection strength is not always the same with the maximum amplitude of the siesmic trace. The reflection strength has its maximum at phase point.

V. CONCLUSION

Hilbert transform techniques plays an important role for generating quadrature trace from an observed real seismic trace. The quadrature trace and the real seismic trace can be combined to yield the reflection strength or the envelope which represents the total energy of the propagating waves. A part from the reflection strength, the instantaneous phase and instantaneous frequency can also be obtained. These in formations are very valuable for detail analysis of complex seismic data.

The Hilbert transform technique can be executed more efficiently using the Fast Fourier transform algorithm which enables us to obtain the quadrature trace in a very quick manner.

The Hilbert transform can be used to convert a non minimum delay wavelet into its corresponding minimum delay wavelet which has the same magnitude spectral. This convertion is useful to facilitate analysis and processing of complex seismic signal.

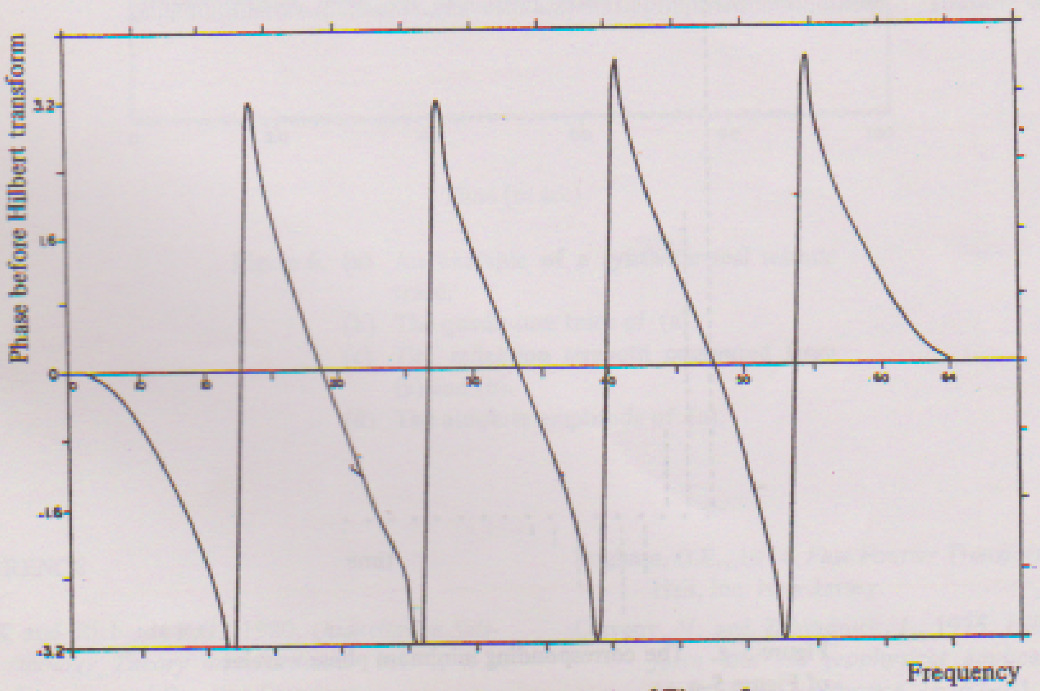

Figure 5–c. The phase spectrum of Figure 5–a.

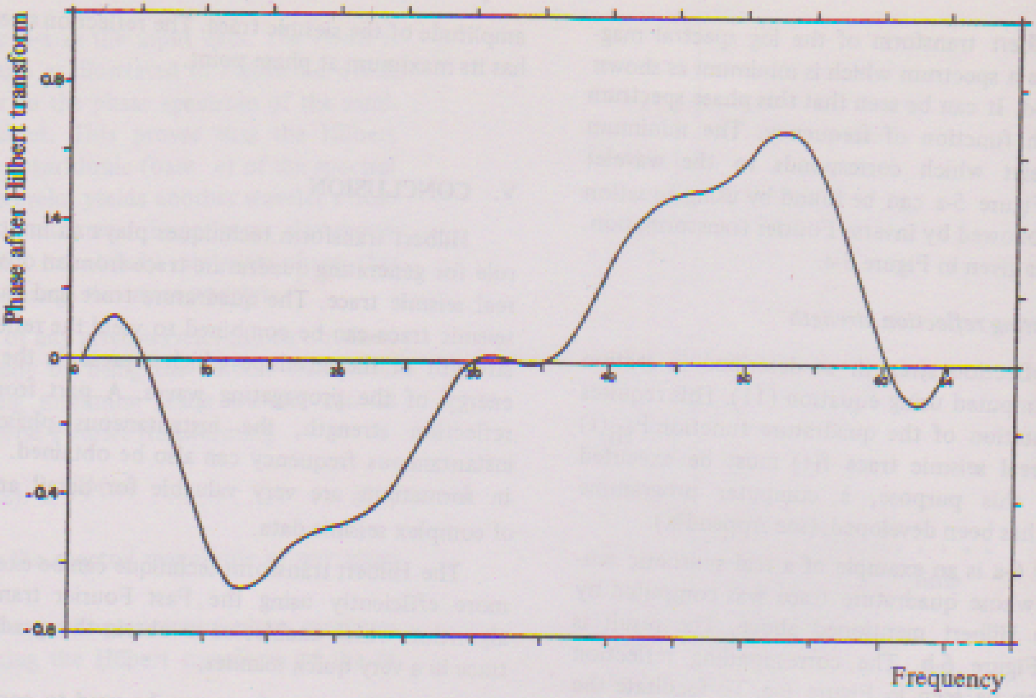

Figure 5–d. The phase spectrum of Figure 5–c, after Hilbert transforming the log spectral magnitude of Figure 5–b.

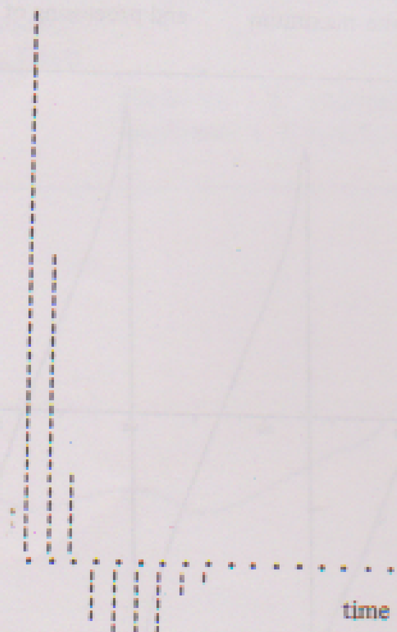

Figure 5–e. The corresponding minimum phase wavelet of Figure 5–a.

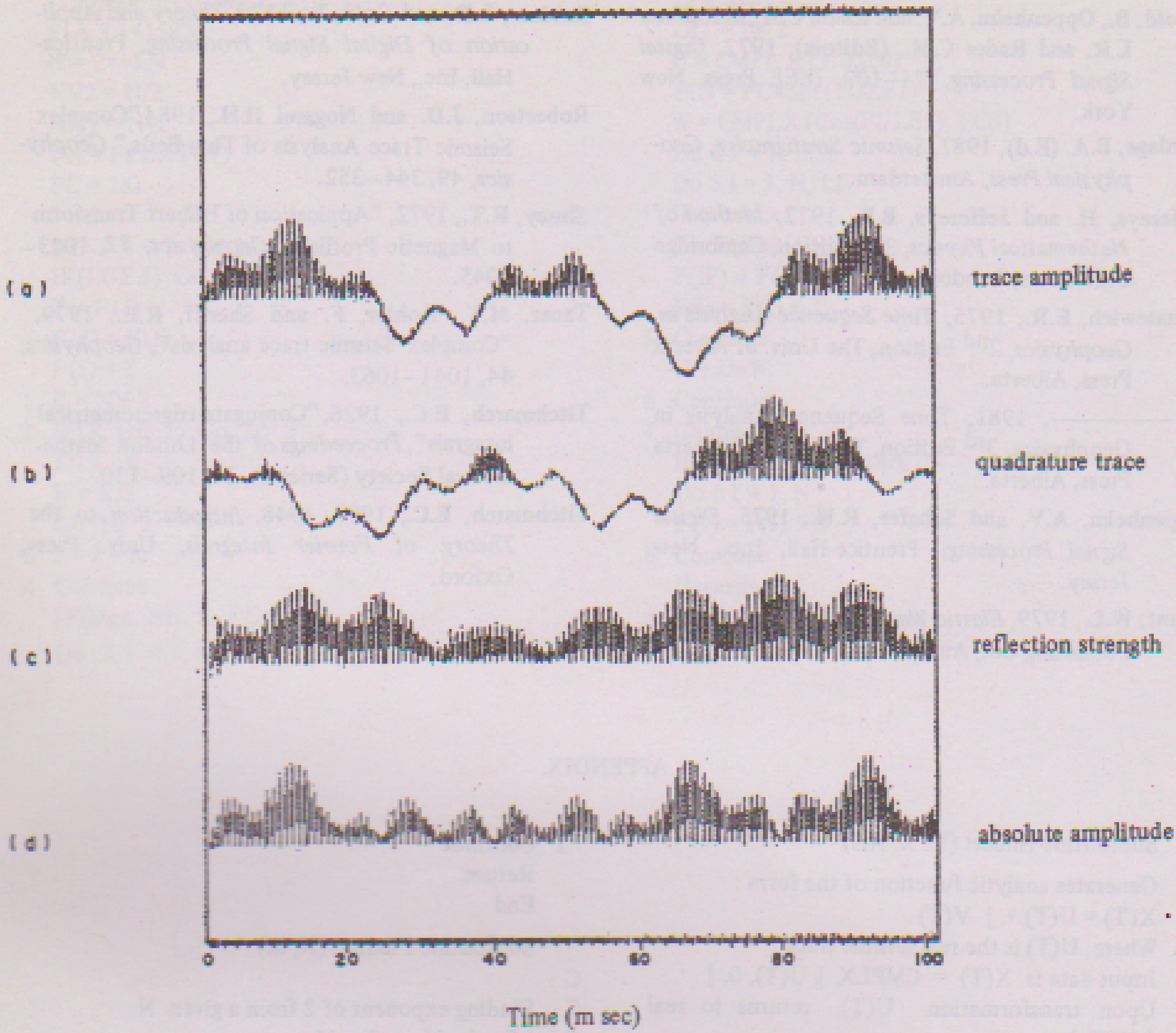

Figure 6. (a) An example of a synthetic real seismic trace.
(b) The quadrature trace of (a).
(c) The reflection strength computed from (a) and (b).
(d) The absolute amplitude of (a).

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APPENDIX.

```
C    Subroutine Hilbert (N, X, XH)
C    Generates analytic function of the form :
C    X(T) = U(T) + j V(T)
C    Where U(T) is the real seismic trace
C    Input data is X(T) = CMPLX [ U(T), 0. ]
C    Upon transformation U(T) returns to real
     [ X(T) ]
C    Hilbert transform of U(T) = V(T) is stored in
     XH(T)
     Dimension X(N), XH(N)
     Complex X
     Call Findnu (N, LN)
     Call FFT (X, -1., LN)
     Do 1 I = 2, N/2
          X (I) = 2. * X (I)
          X (I + N/2) = 0.0
  1  Continue
     X (1+N/2) = 0.0
     Call FFT (X, + 1., LN)
     Do 2 I = 1, N
          XH (I) = -AIMAG (X(I))
  2  Continue
     Return
     End
     Subroutine Findnu (N, M)
C
C    Finding exponent of 2 from a given N
C    I.E.      N = 2 **M
C
     Do 10 I = 1,12
          IF(N - 2**I) 10, 5, 10
  5       M = I
          Goto 20
 10 Continue
 20 Return
    End
C234567
        Subroutine FFT (F, Sign, LN)
C
C       Sign = -1.  For direct transformation
C       Sign = +1.  For invers transformation
C       2**LN = N
```

```
      Complex  F(1024), J, W, T
      PI = 3.141593
      N = 2**LN
      NV2 = N/2
      NM1 = N-1
      FN = FLDAT(N)
      FC = 1.0
      J = 1
      DO  4  I = 1, NM 1
      IF(I.GE.J)  Go  To  1
      T = F(J)
      F(J) = F(I)
      F(I) = T
1     K = NV 2
2     IF(K. GE. J)  Go  To  3
      J = J - K
      K = K/2
      Go  To  2
3     J = J + K
4     Continue
      IF(Sign. NE. 1)  FC = -FC
      Do  7  L = 1, LN
      Le = 2**L
      Le 1 = Le/2
      U = CMPL X (1.0,0.0)
      FCS = FC*Sin (PI/LE1)
      W = CMPLX (Cos(PI/LE1), FCS)
      Do 6 J = 1, LE1
      Do 5 I = J, N, LE
      IP = I + Le1
      T = F(IP)* U
      F(IP) = F(i) - T
      F(I) = F(I) + T
5     Continue
      U = U*W
6     Continue
7     Continue
      If(sign. eq. -1.) Return
      Do 8 I = 1, N
      F(I) = F(I)/FN
8     Continue
      Return
      End
```